

Fractional reaction diffusion along flow lines

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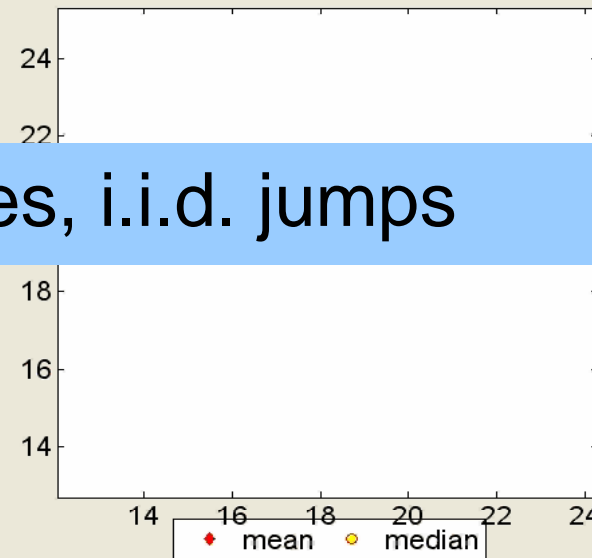
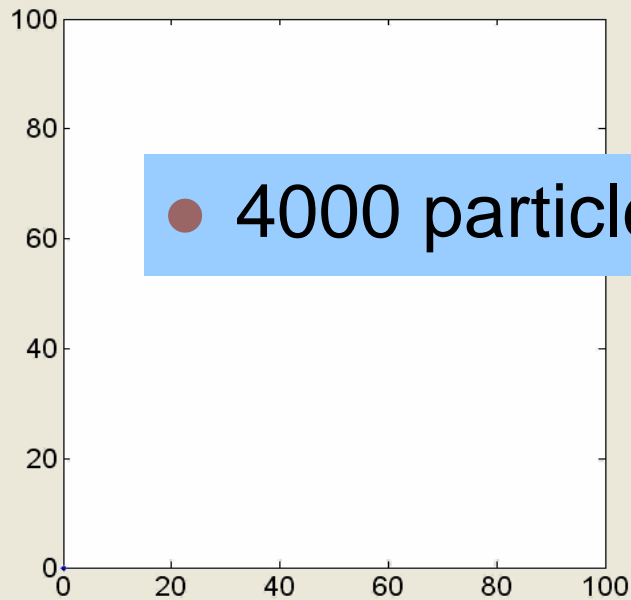
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Complexity and the fractional model

- Complex = Random ?!
 - More or less (Bayesian Framework)...
- Even if individual realisations are random much can be said on aggregate (mean-field model)
- Fractional model is based on sums of random variables modelling complex movement of objects.

The Central Limit Theorem

100



Go

Stop

Reset

Number of particles

4000

Extremes omitted

20

0

0

100

X-jumps

Uniform [0,1]

Y-jumps

Uniform [0,1]

Waiting Time

1 time unit

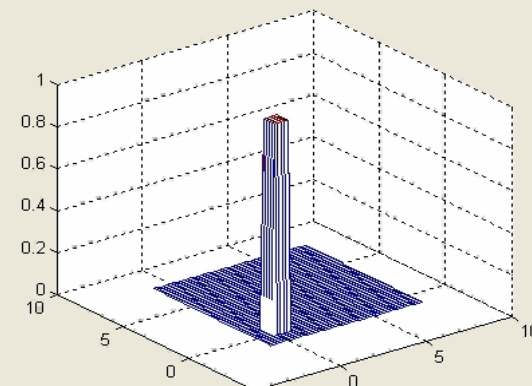
jumps

Power:

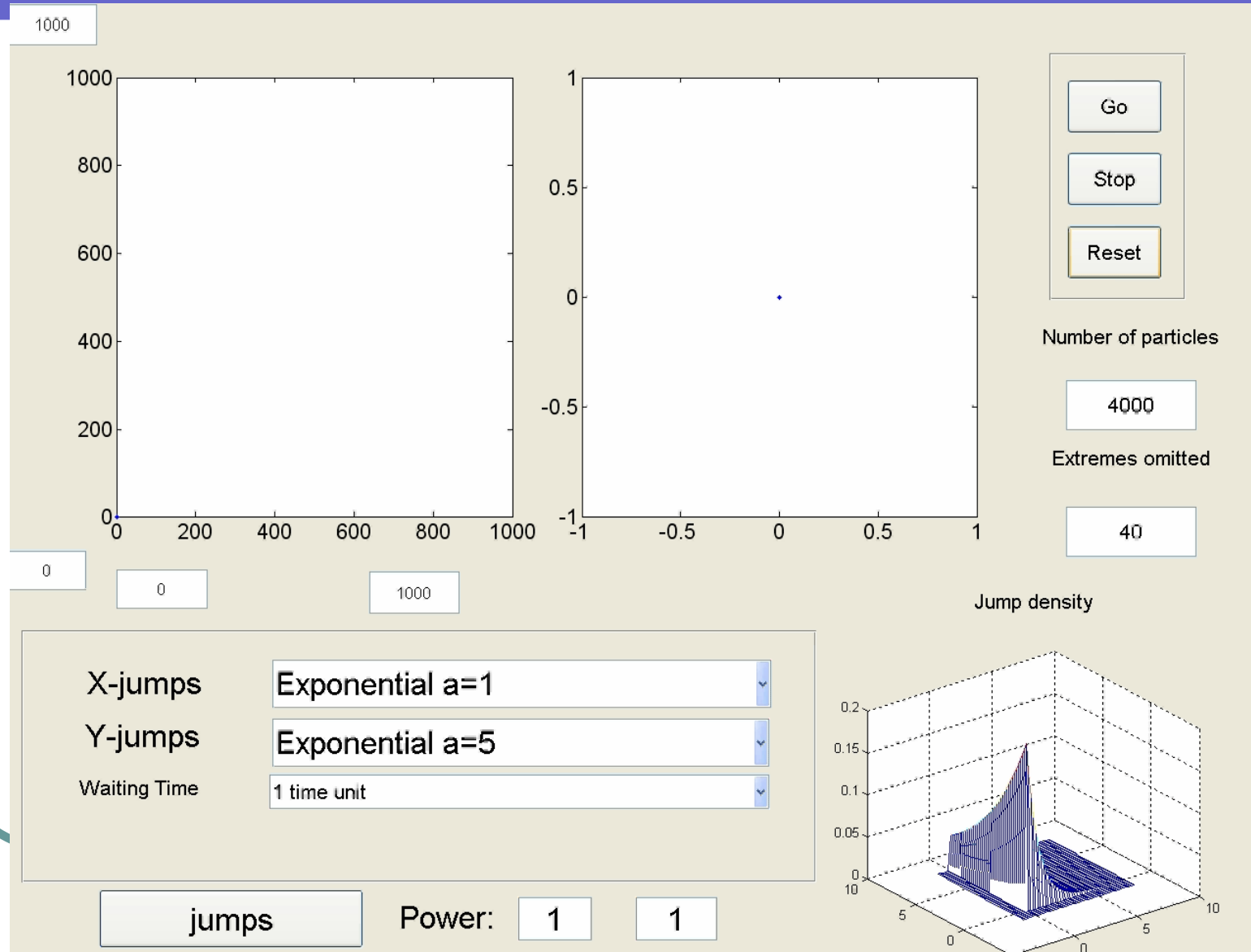
1

1

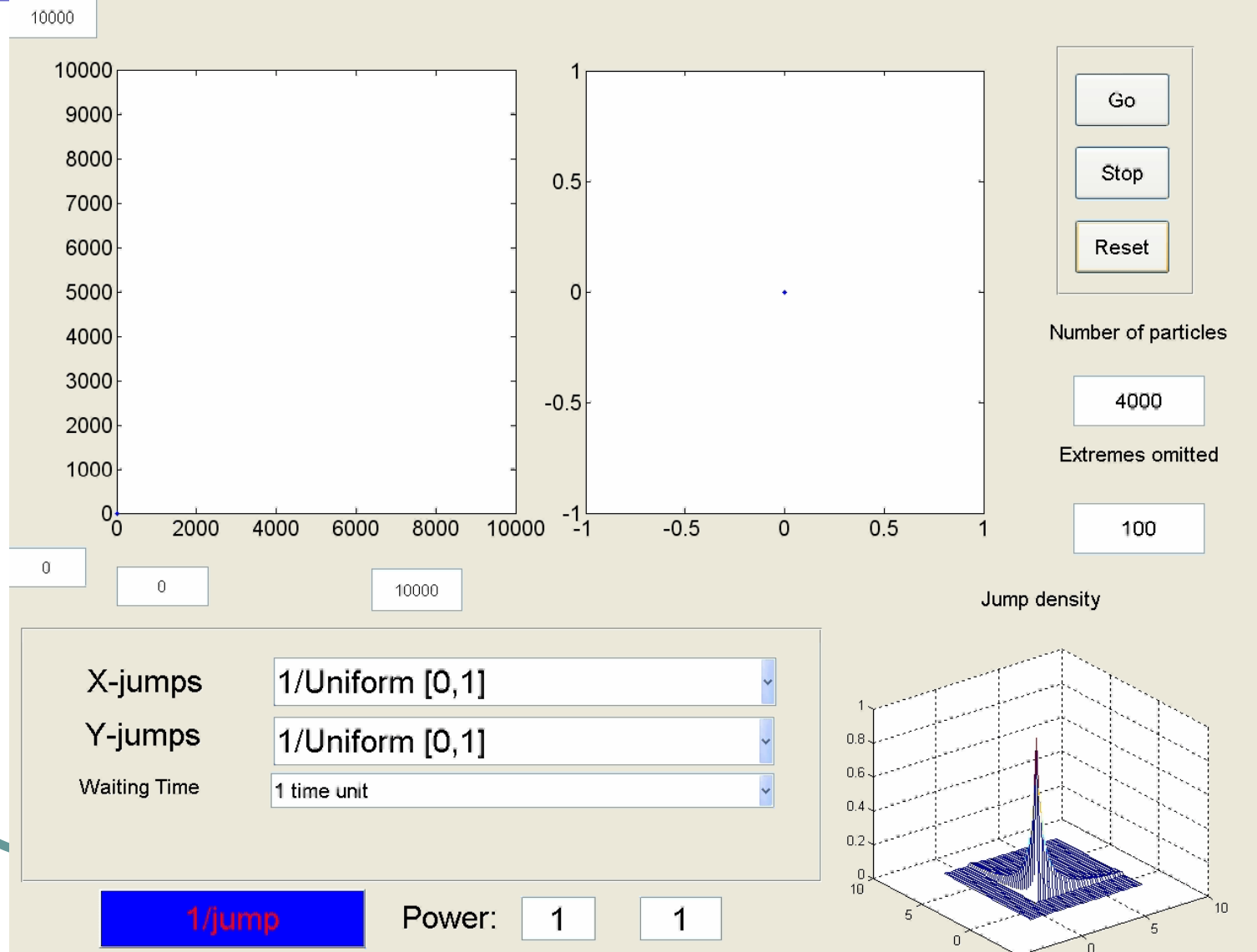
Jump density



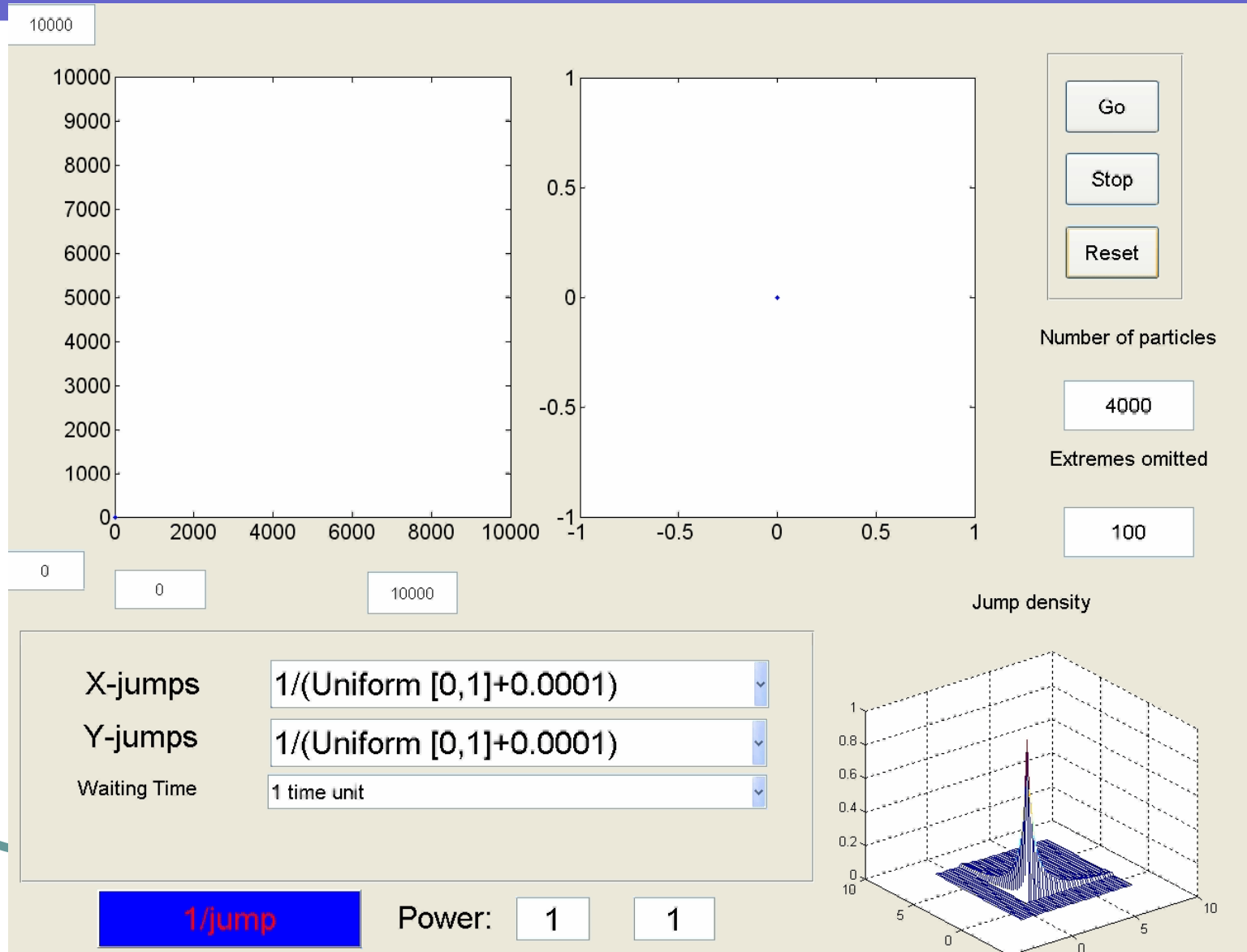
The CLT, exponential



The CLT, power-law

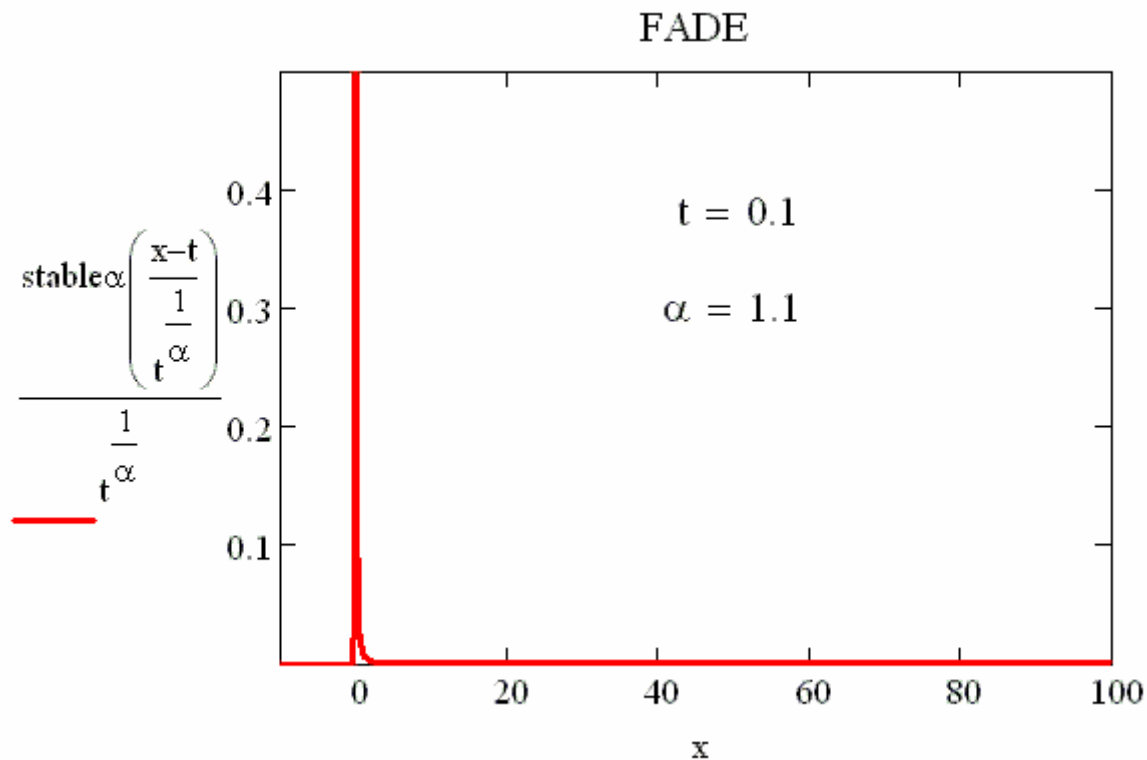


The CLT, truncated power-law

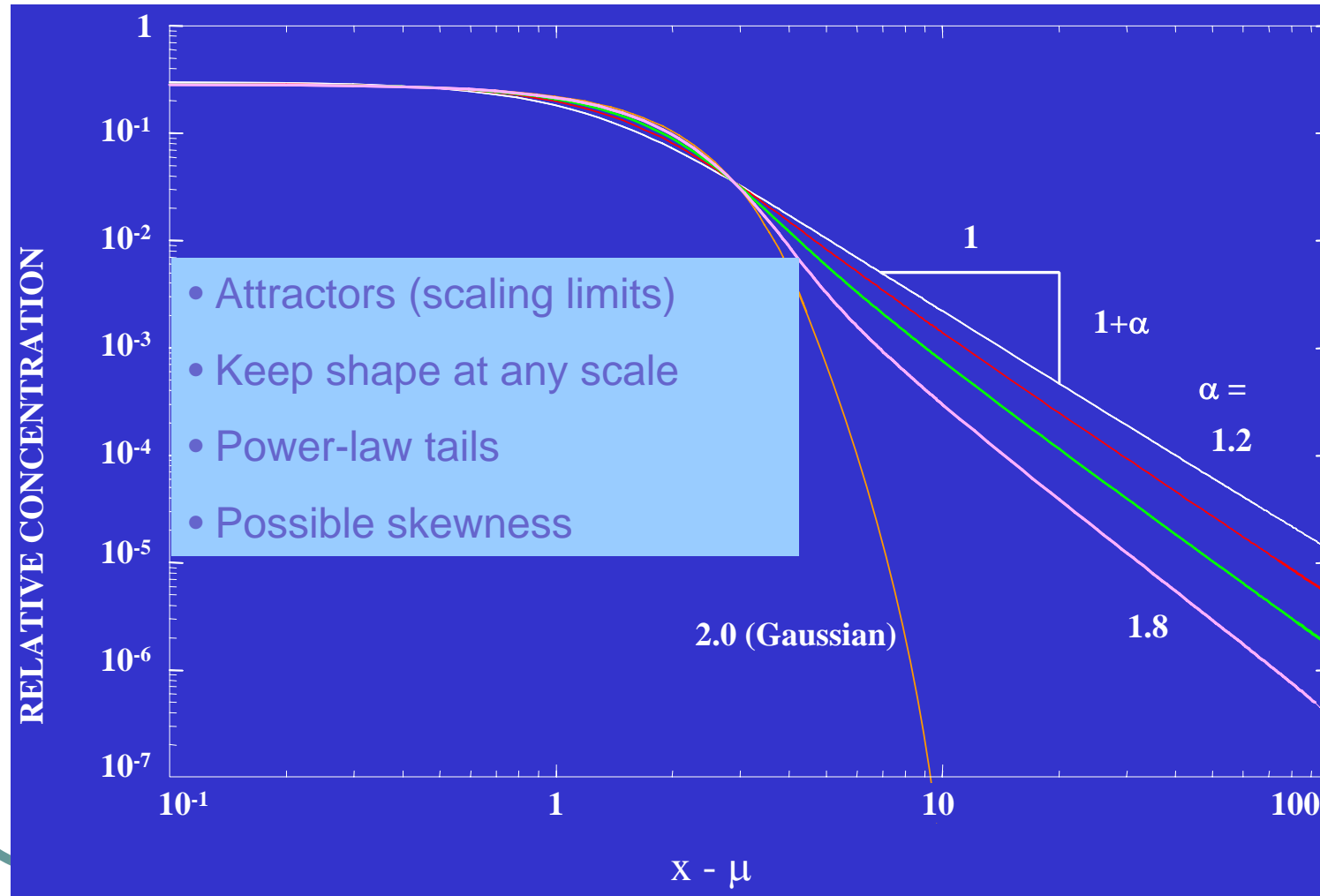


The FADE

$$\frac{\partial}{\partial t} u(t, x) = -v \frac{\partial}{\partial x} u(t, x) + D \left(\frac{\partial}{\partial x} \right)^\alpha u(t, x)$$



Stable densities

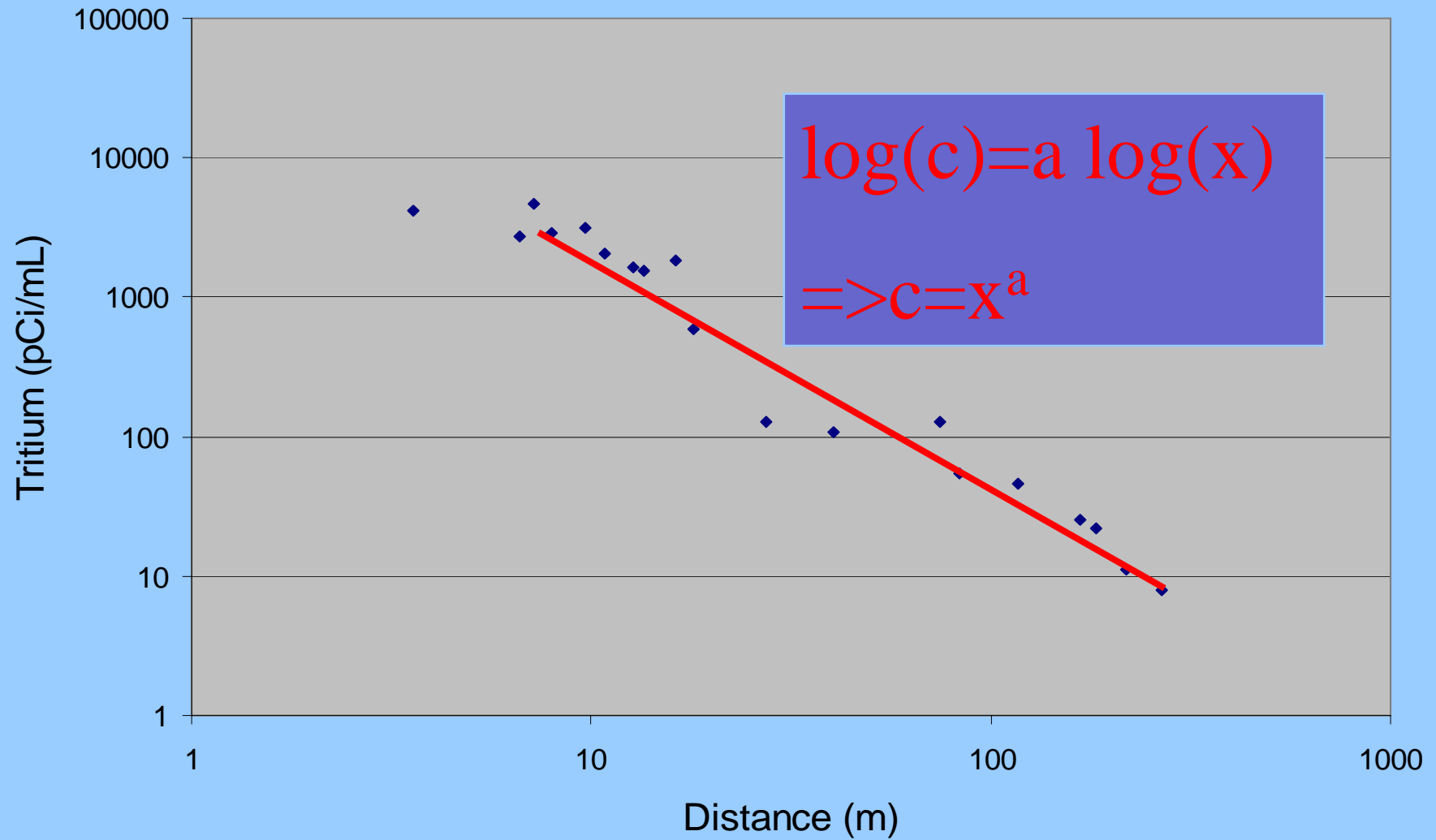


Example: Solute transport in Groundwater

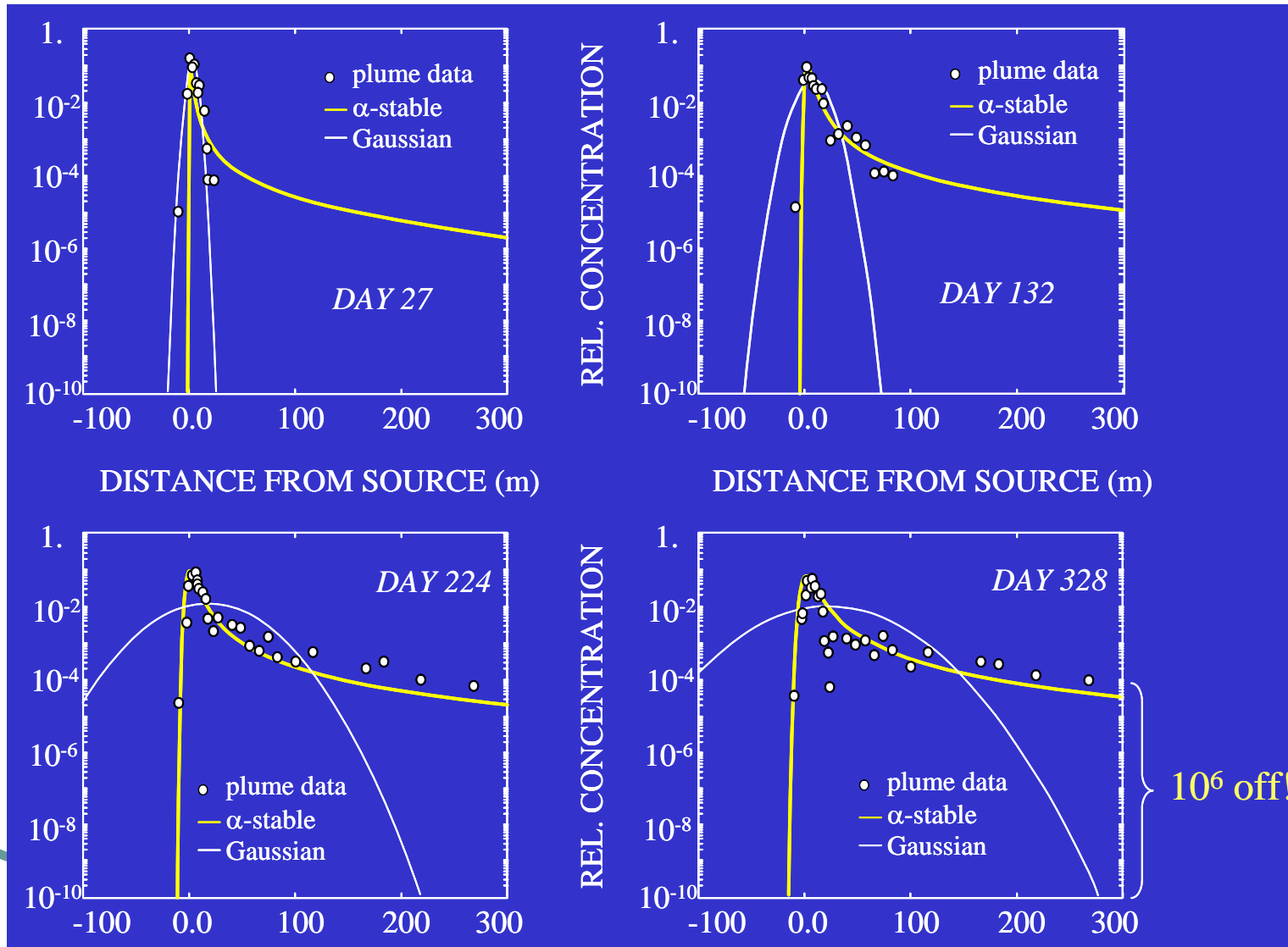


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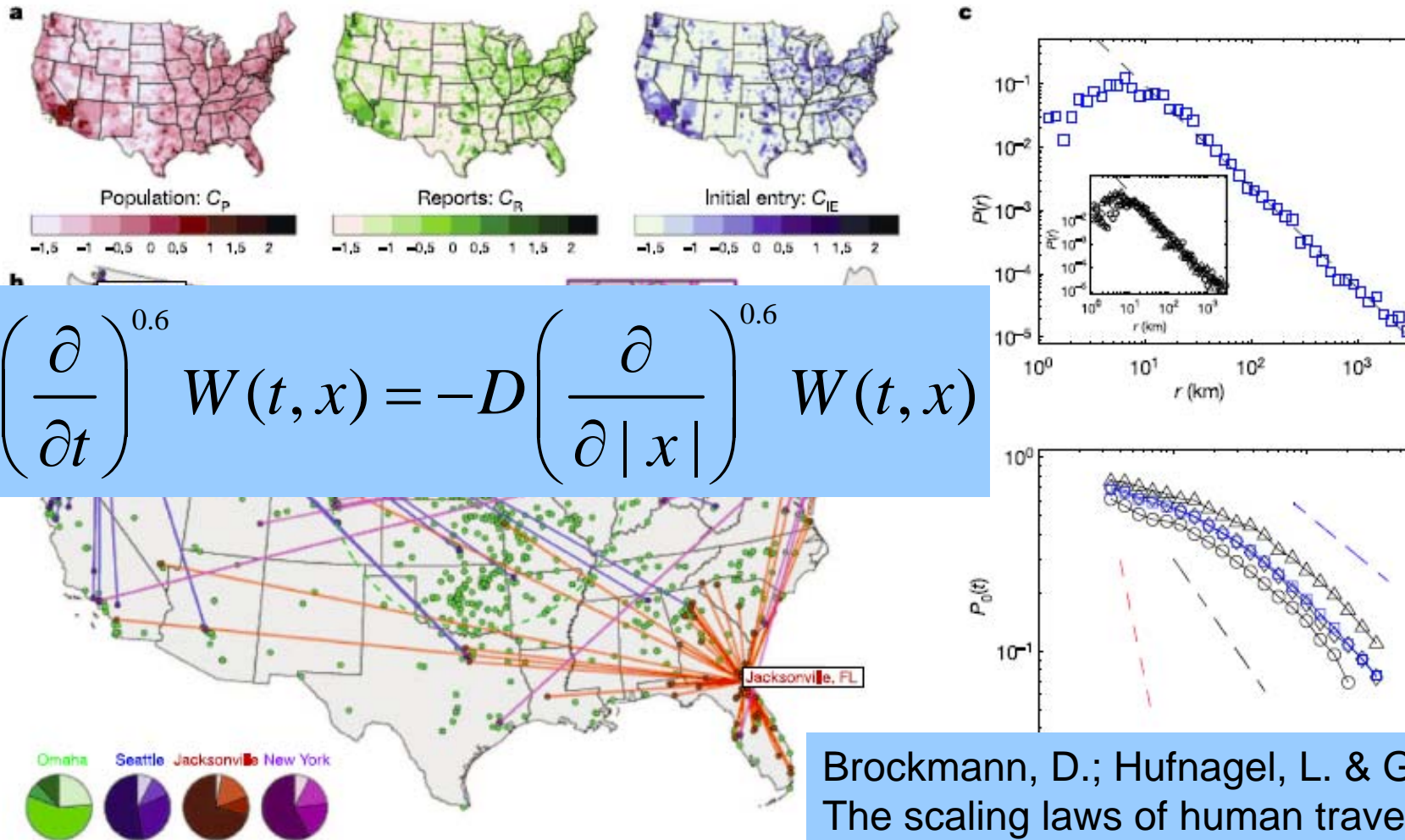
Day 328 Columbus AFB (MS)



MADE site semilog



Where is George?



$$\left(\frac{\partial}{\partial t}\right)^{0.6} W(t, x) = -D \left(\frac{\partial}{\partial |x|}\right)^{0.6} W(t, x)$$

Brockmann, D.; Hufnagel, L. & Geisel, T. The scaling laws of human travel *Nature*, 2006, 439, 462

Figure 1 | Dispersal of bank notes and humans on geographical scales. dispersal

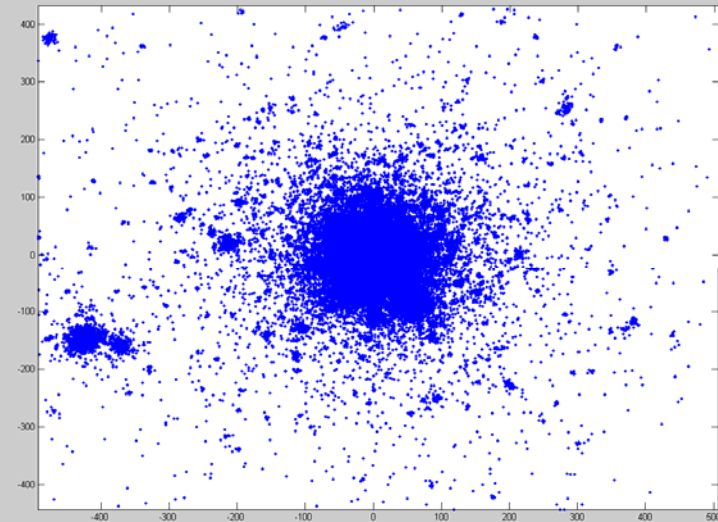
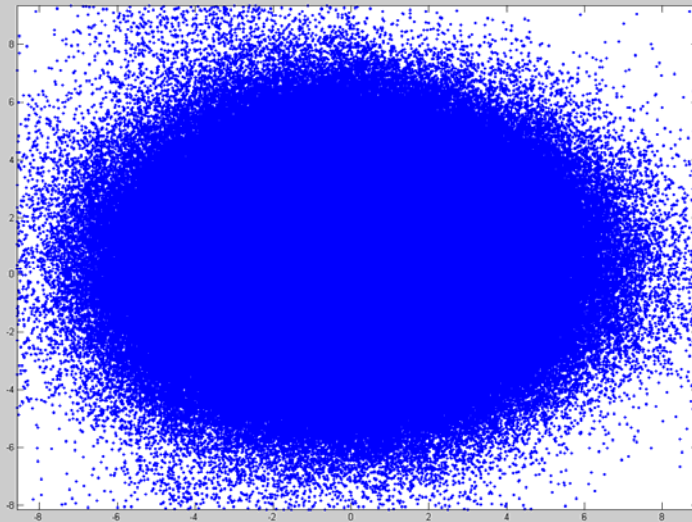
Reaction-Dispersion

- Epidemiology
- Invasion of species
 - Dispersal is often heavy-tailed
- ...
- Add reaction term :

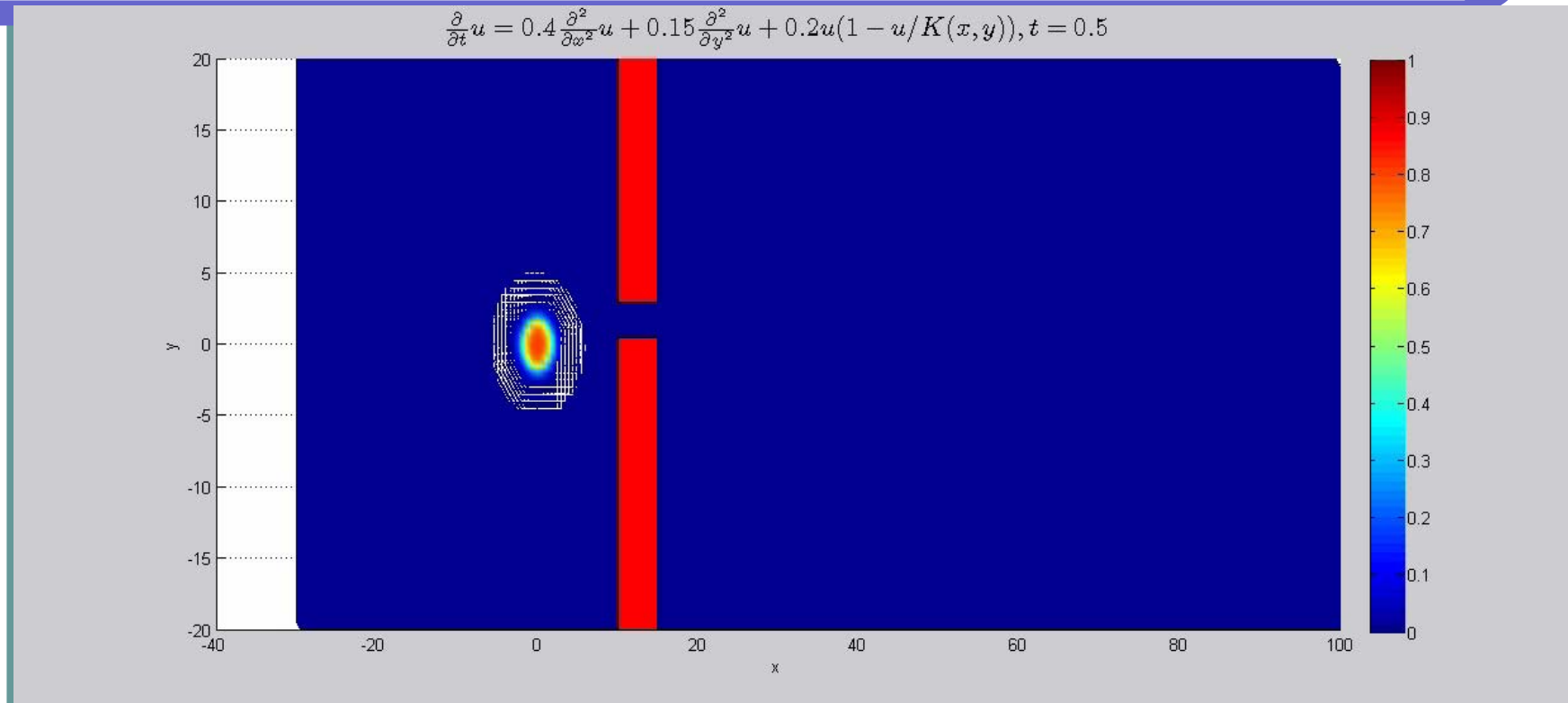
$$\frac{\partial}{\partial t} C(t, x) = -D(-\Delta)^{\alpha/2} C(x, t) + f(C(x, t))$$

Reproduction-Dispersal

- Monte-Carlo, heavy tailed dispersal
 - 6 generations @ 10 children, $\Pr\{\text{Jump} > x\} = x^{-\alpha}$
 - $\alpha > 2$
 - $\alpha < 2$

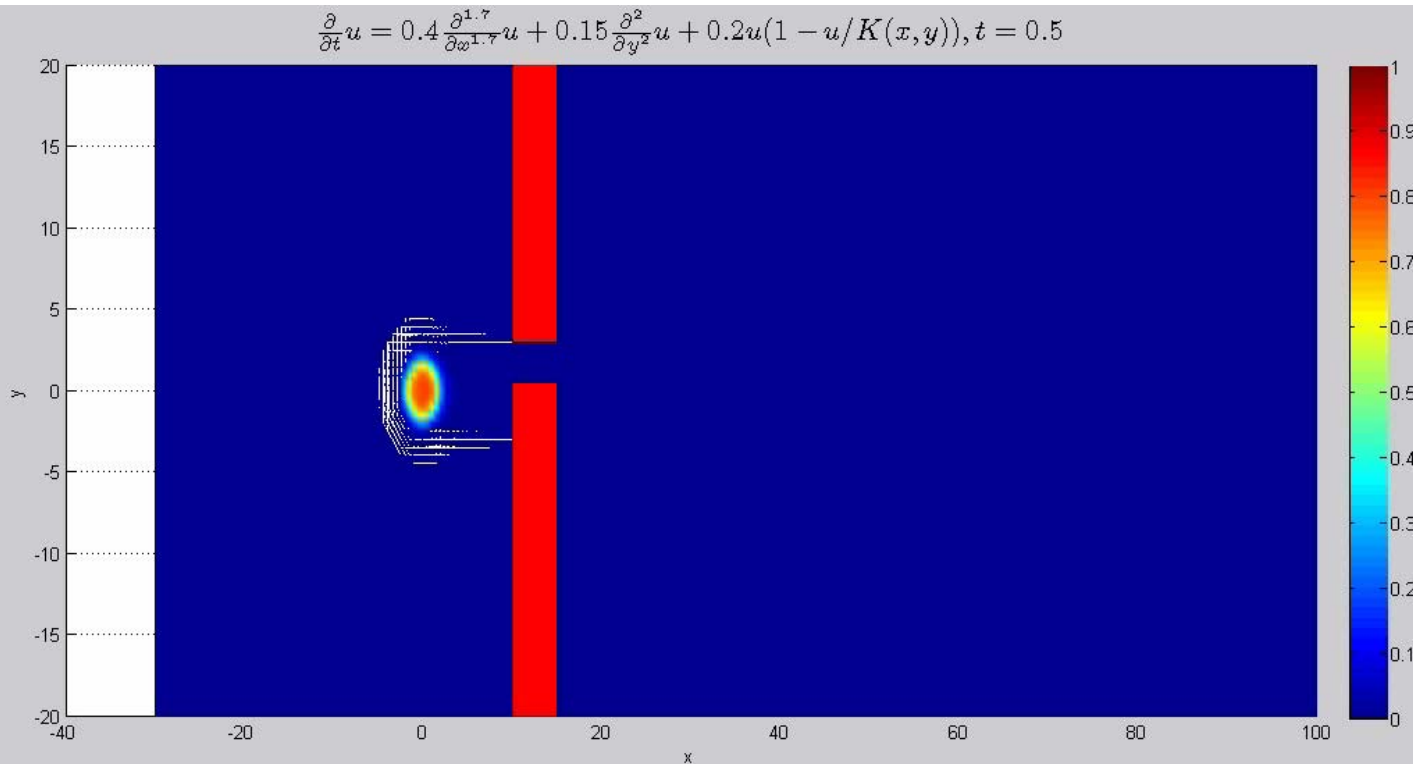


Classical Fisher equation



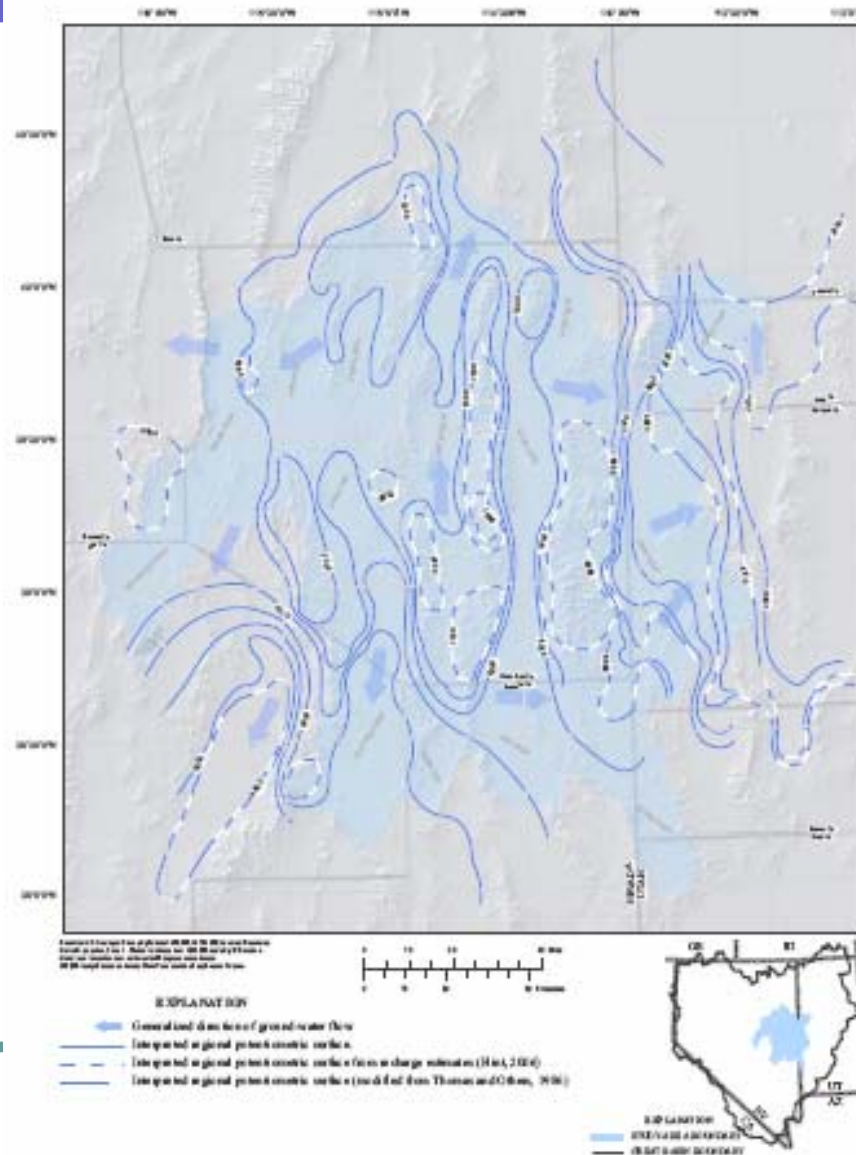
$$\frac{\partial}{\partial t} u = D \Delta u + ru(1 - u / K)$$

Fractional Fisher equation



$$\frac{\partial}{\partial t} u = D_1 \frac{\partial^\alpha}{\partial x^\alpha} u + D_2 \frac{\partial^2}{\partial y^2} u + ru(1 - u/K)$$

Movement on a regional scale



Postulates

- Long jumps should follow flow lines
- Long jump in a slow region is more improbable than in a fast region
- Locally keep one-dimensional advection and fractional dispersion with maybe some lateral diffusion.

The Idea

- Instead of jump size, randomise velocity, or time available (Bochner subordination).

$$\frac{\partial}{\partial t} u(t, \mathbf{x}) = -\nabla \mathbf{g}(\mathbf{x}) u(t, \mathbf{x})$$

$$C(t, x) = u(s_t, x); s_t \text{ At } + t^{1/\alpha} L(0, D)$$

- In operator semi-group notation with initial distribution f and flow-group G :

$$C(t) f = \int_{-\infty}^{\infty} g_t(s) G(s) f ds$$

Anomalous dispersion along flow-lines

- Main dispersal pathways (not necessarily straight) might be identifiable:
 - Randomise velocity (time) according to 1D fractional model.
 - Leads to fractional powers $1 \leq \alpha \leq 2$ of transport operator A (generator of flow group)
 - Solve by operator-split methods

$$\frac{\partial}{\partial t} u = A^\alpha u + ru(1 - u / K)$$

Conclusions

- Parsimonious linear model capturing (mean-field) anomalous dispersion.
- Robust as based on limit theorem.
- Extends classical models by simply replacing diffusion operator with fractional diffusion operator
- Predicts accelerating wave-speed, power-law tails, etc...

References

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Subordinated groups of linear operators: properties via the transference principle and the related unbounded operational calculus
Submitted.