Fractional reaction diffusion along flow lines

Boris Baeumer

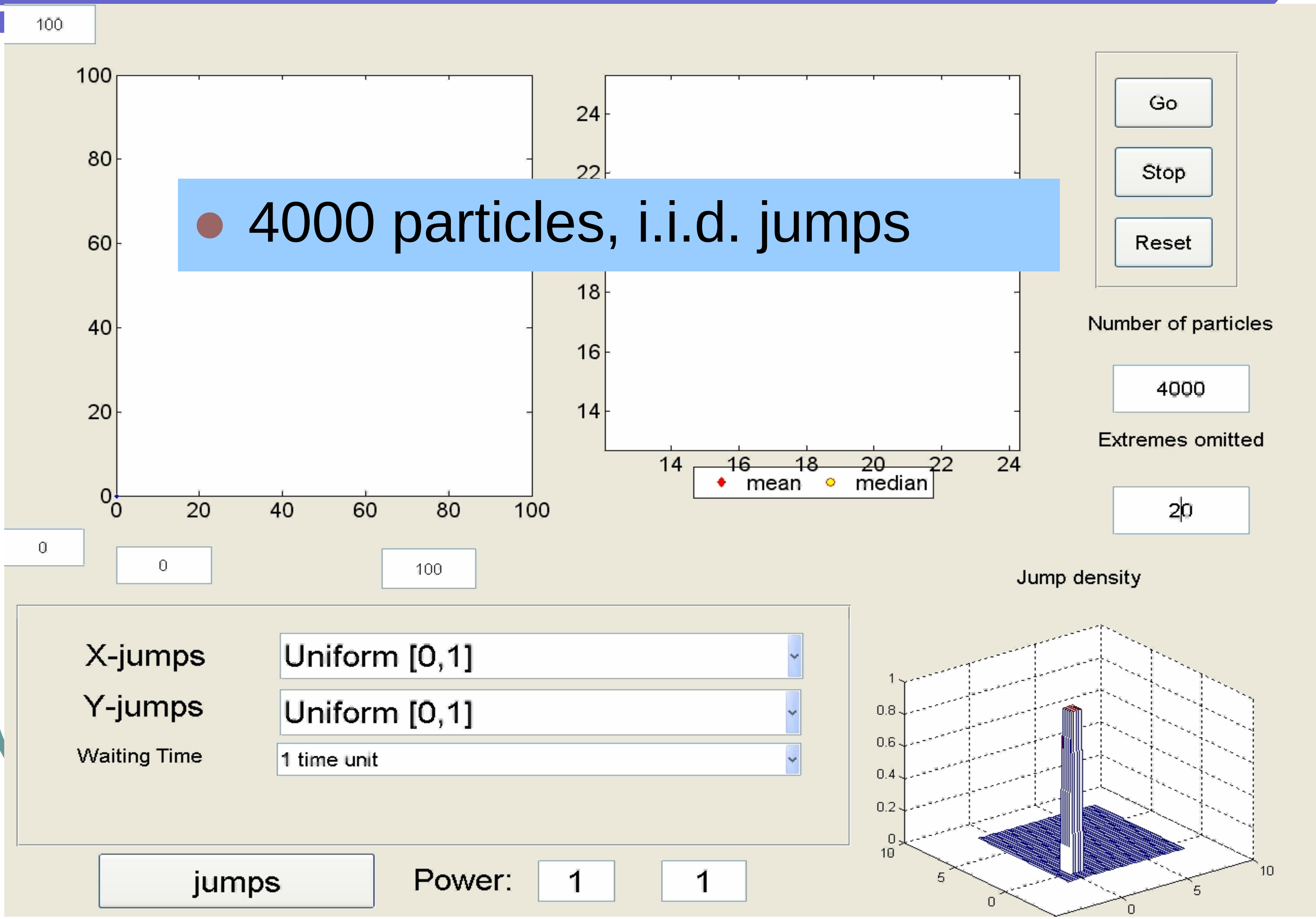
University of Otago

Complexity and the fractional model

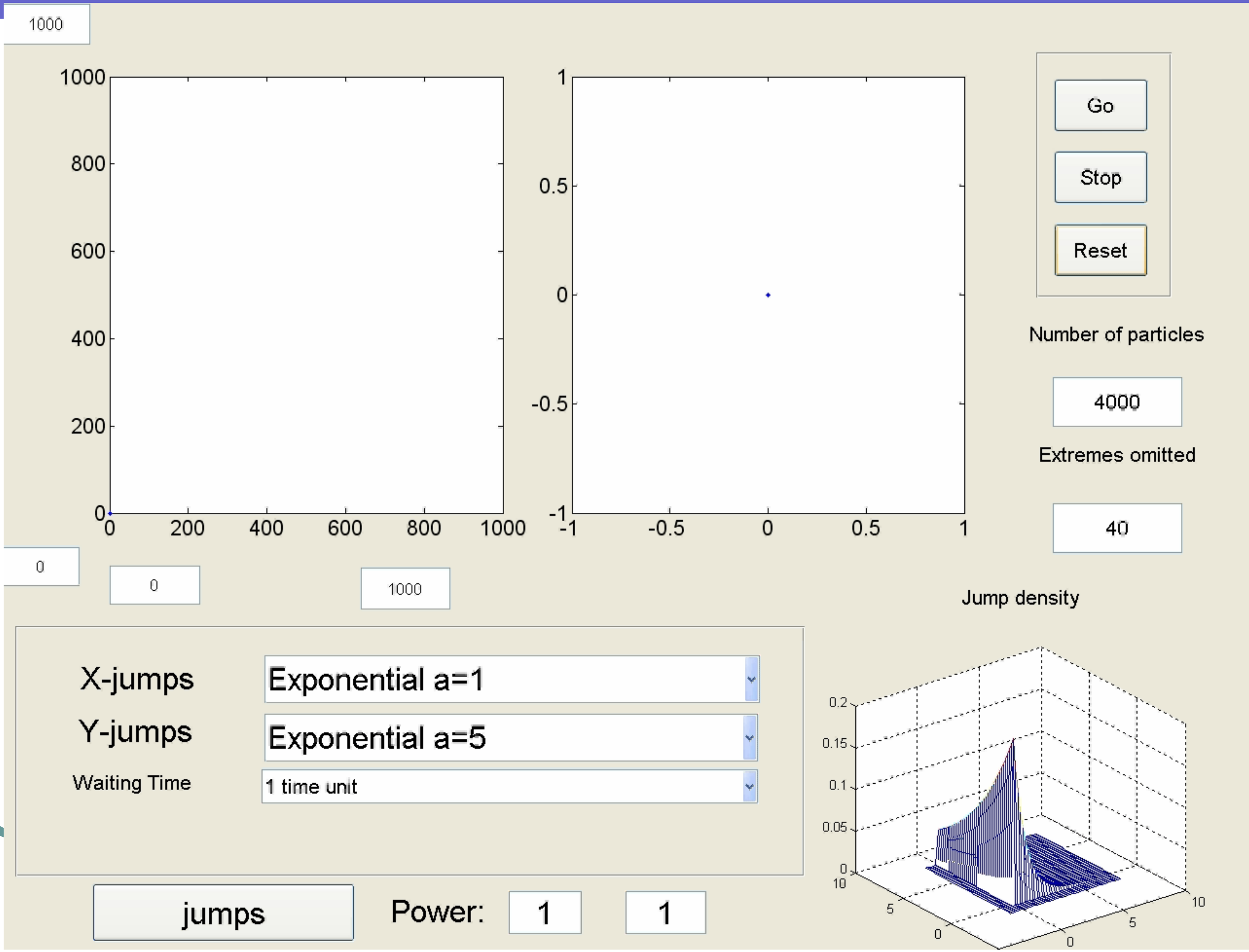
- Complex = Random ?!
  - More or less (Bayesian Framework)…
- Even if individual realisations are random much can be said on aggregate (mean-field model)
- Fractional model is based on sums of random variables modelling complex movement of objects.

The Central Limit Theorem

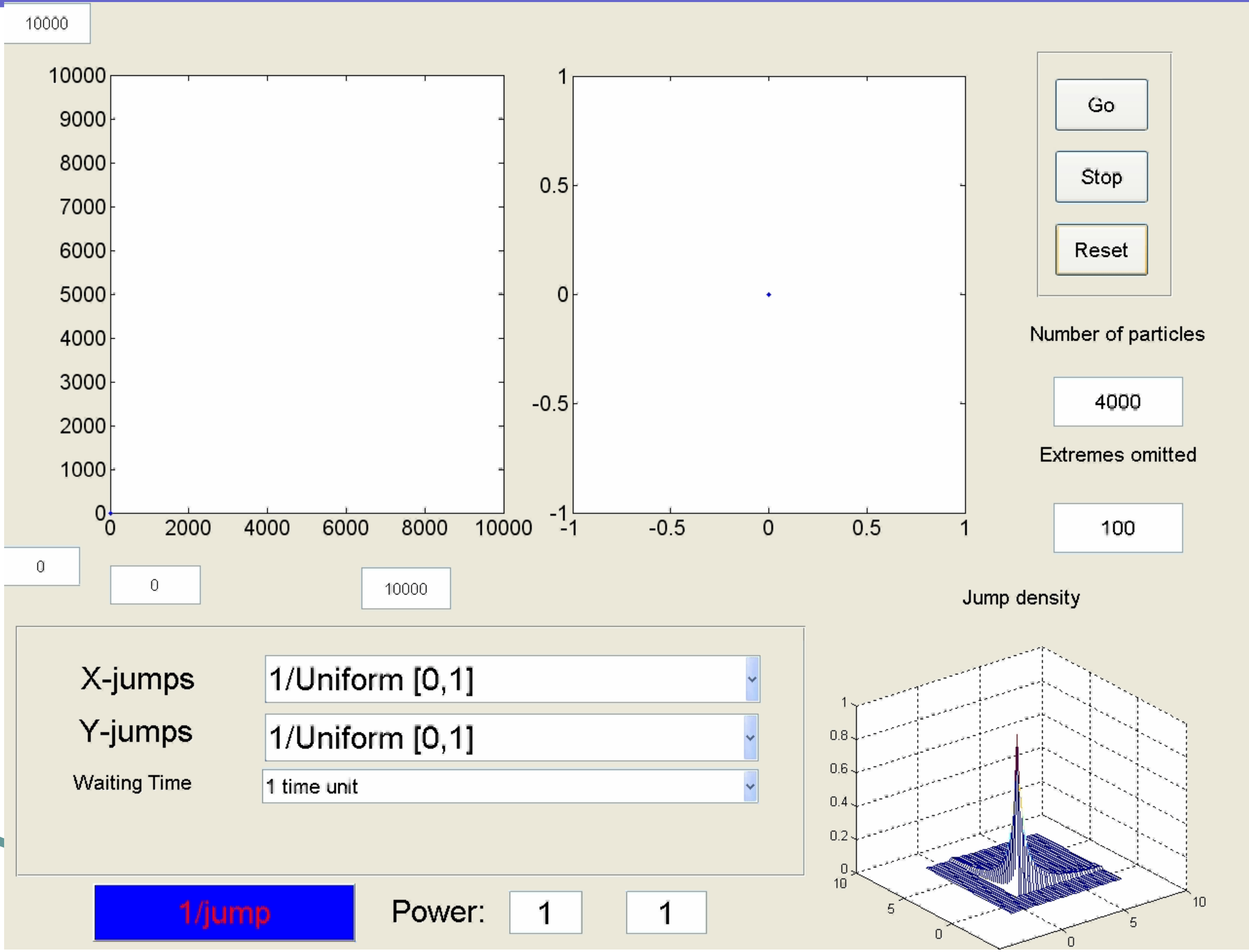
- 4000 particles, i.i.d. jumps



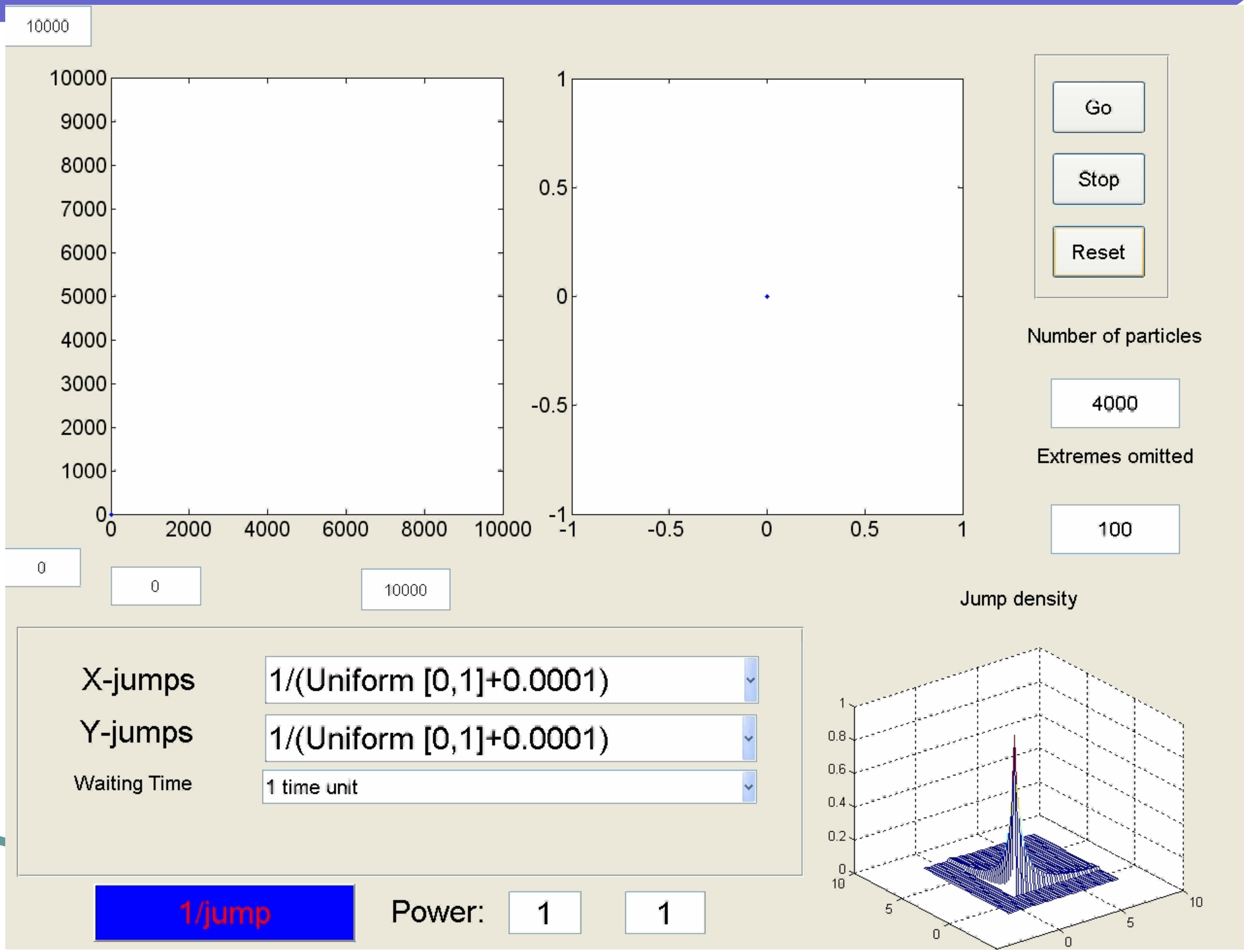
The CLT, exponential



The CLT, power-law

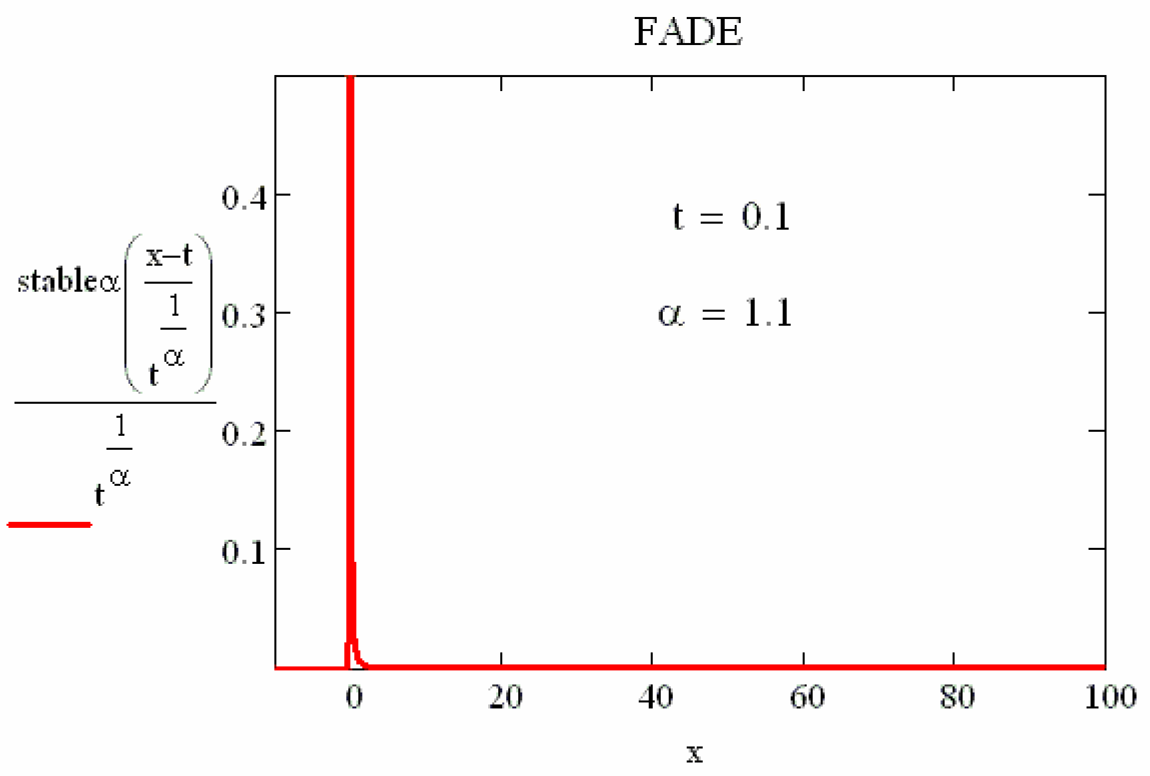


The CLT, truncated power-law

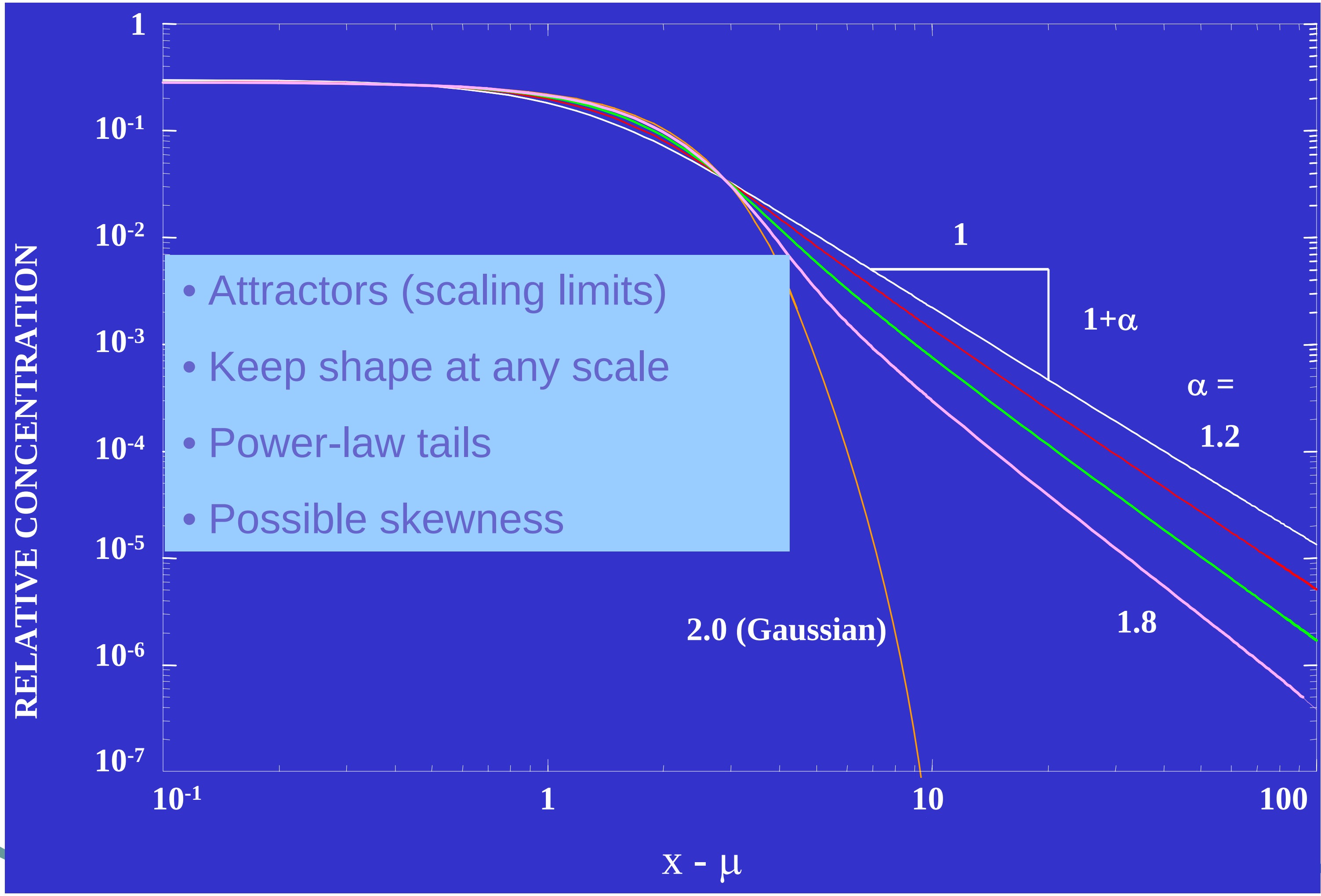


The FADE

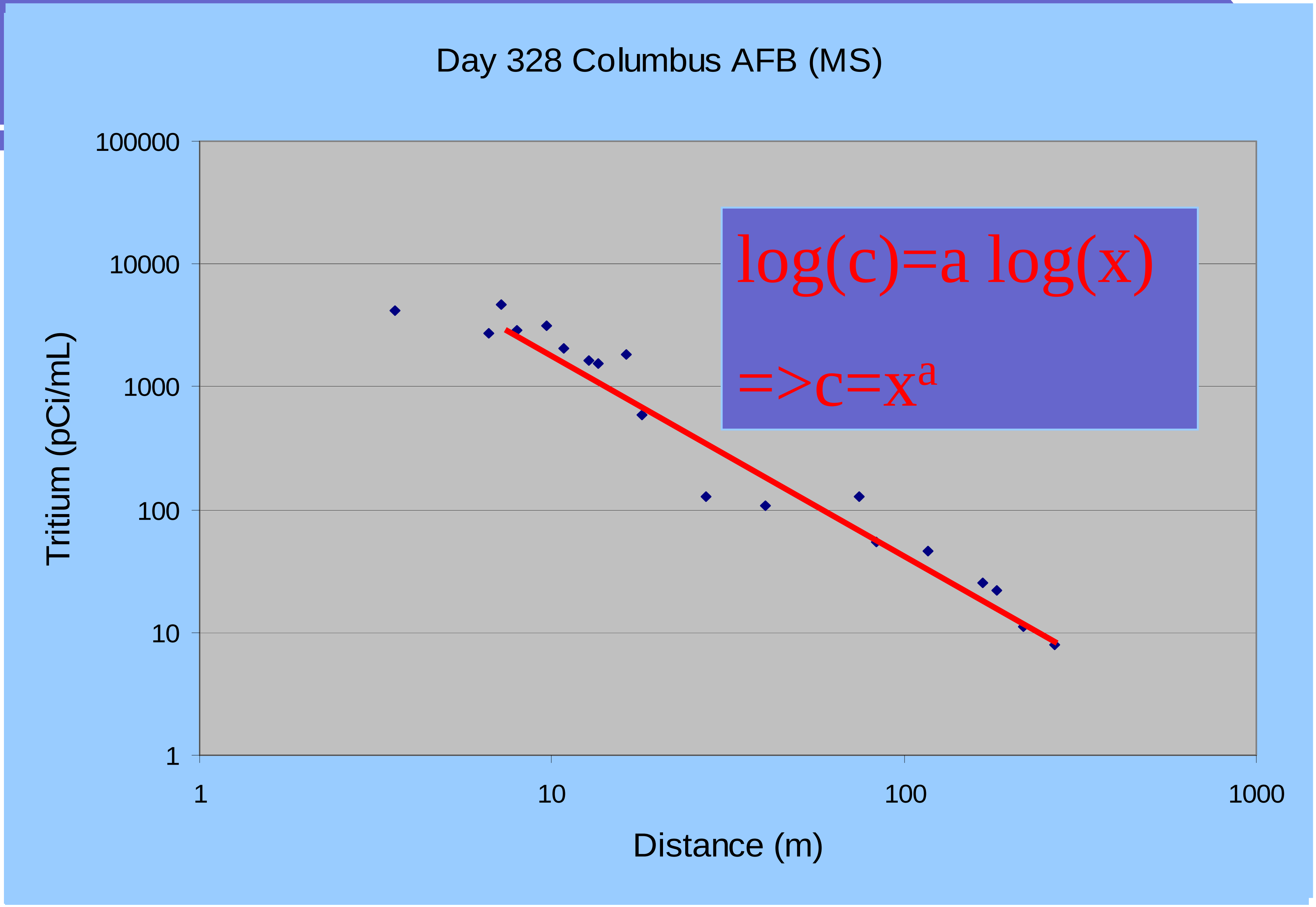
$$\frac{\partial}{\partial t}u(t,x) = -v\frac{\partial}{\partial x}u(t,x) + D\left(\frac{\partial}{\partial x}\right)^{\alpha}u(t,x)$$



Stable densities

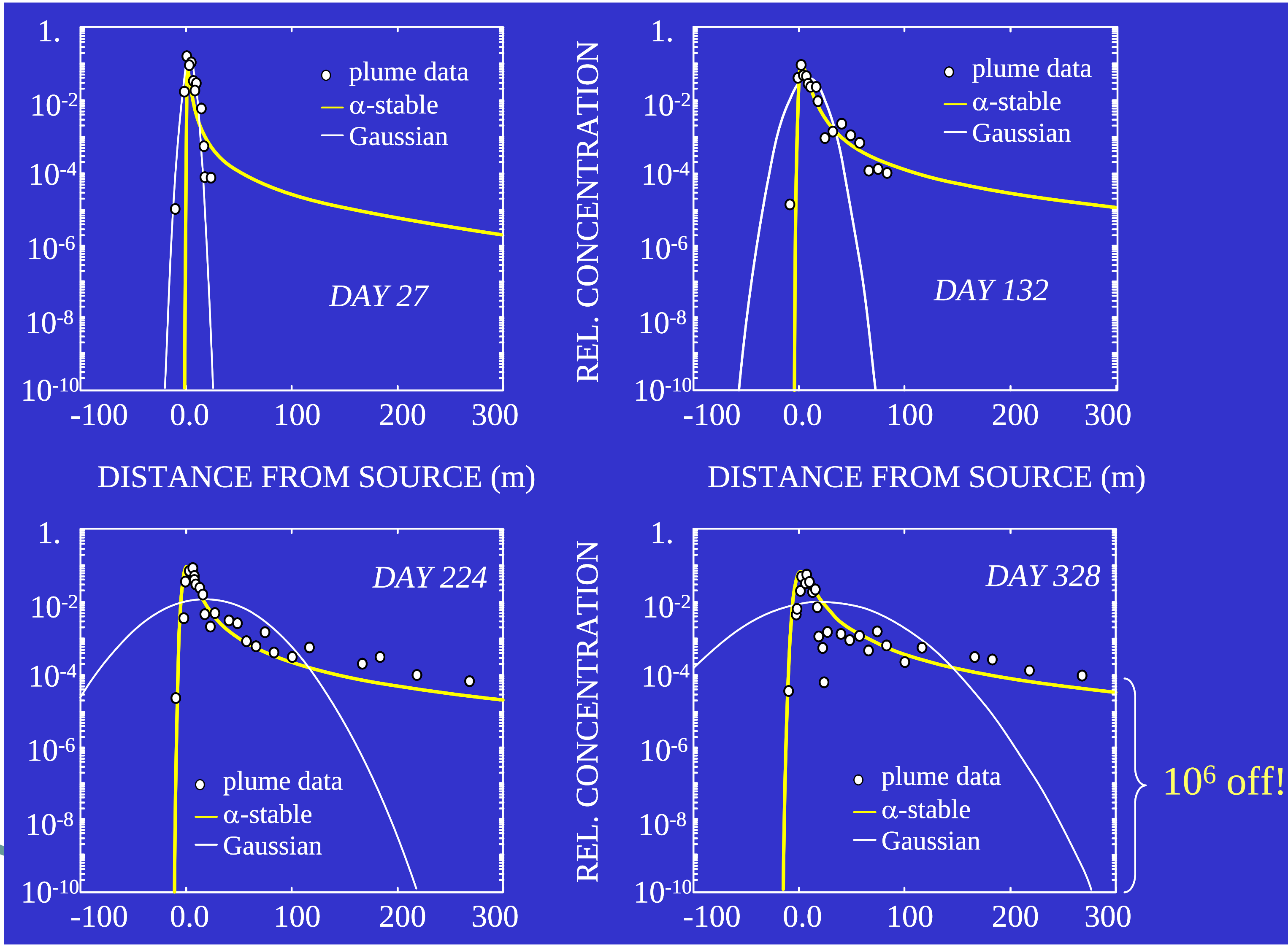


Example: Solute transport in Groundwater

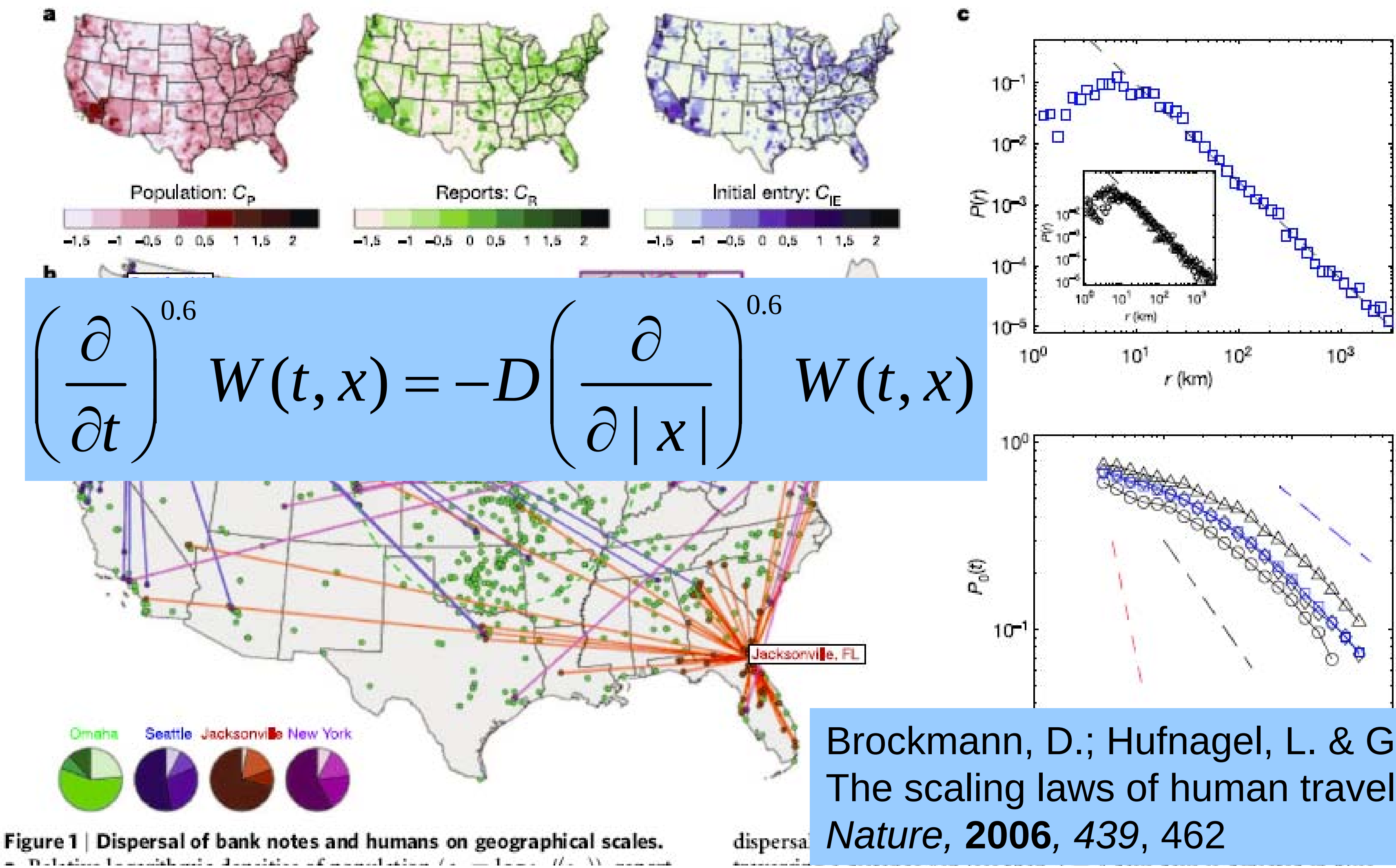


Day 328 Columbus AFB (MS)
100000
10000
1000
100
10
1
Tritium (pCi/mL)
1
10
100
1000
Distance (m)
log(c)=a log(x)
=>c=x^a

MADE site semilog



Where is George?



$$\left(\frac{\partial}{\partial t}\right)^{0.6} W(t,x) = -D\left(\frac{\partial}{\partial |x|}\right)^{0.6} W(t,x)$$

Brockmann, D.; Hufnagel, L. & Geisel, T.
The scaling laws of human travel
Nature, **2006**, *439*, 462

Reaction-Dispersion

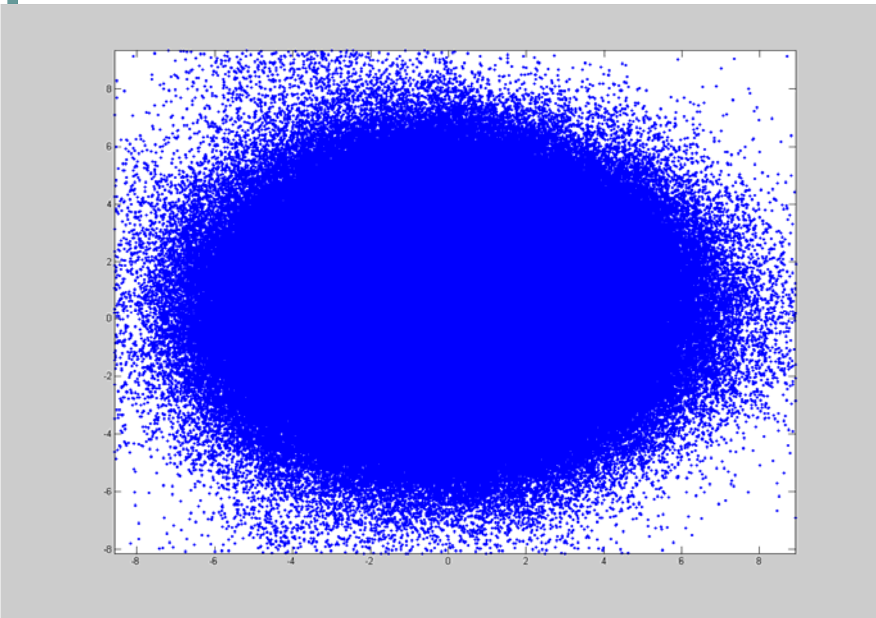
- Epidemiology
- Invasion of species
  - Dispersal is often heavy-tailed
- …
- Add reaction term :

$$\frac{\partial}{\partial t} C(t,x) = -D(-\Delta)^{\alpha/2} C(x,t) + f(C(x,t))$$

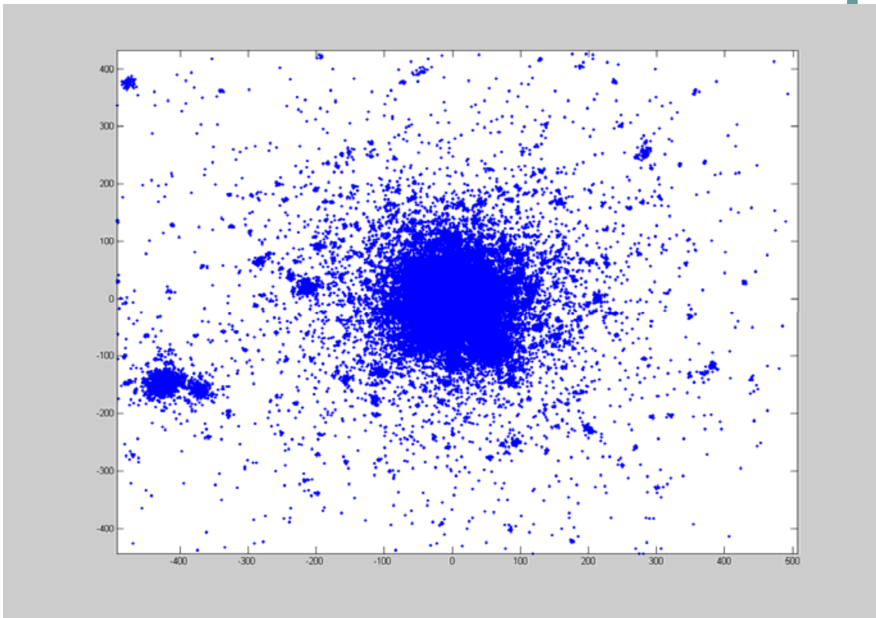
Reproduction-Dispersal

- Monte-Carlo, heavy tailed dispersal
  - 6 generations @ 10 children, $\Pr\{\text{Jump} > x\} = x^{-\alpha}$

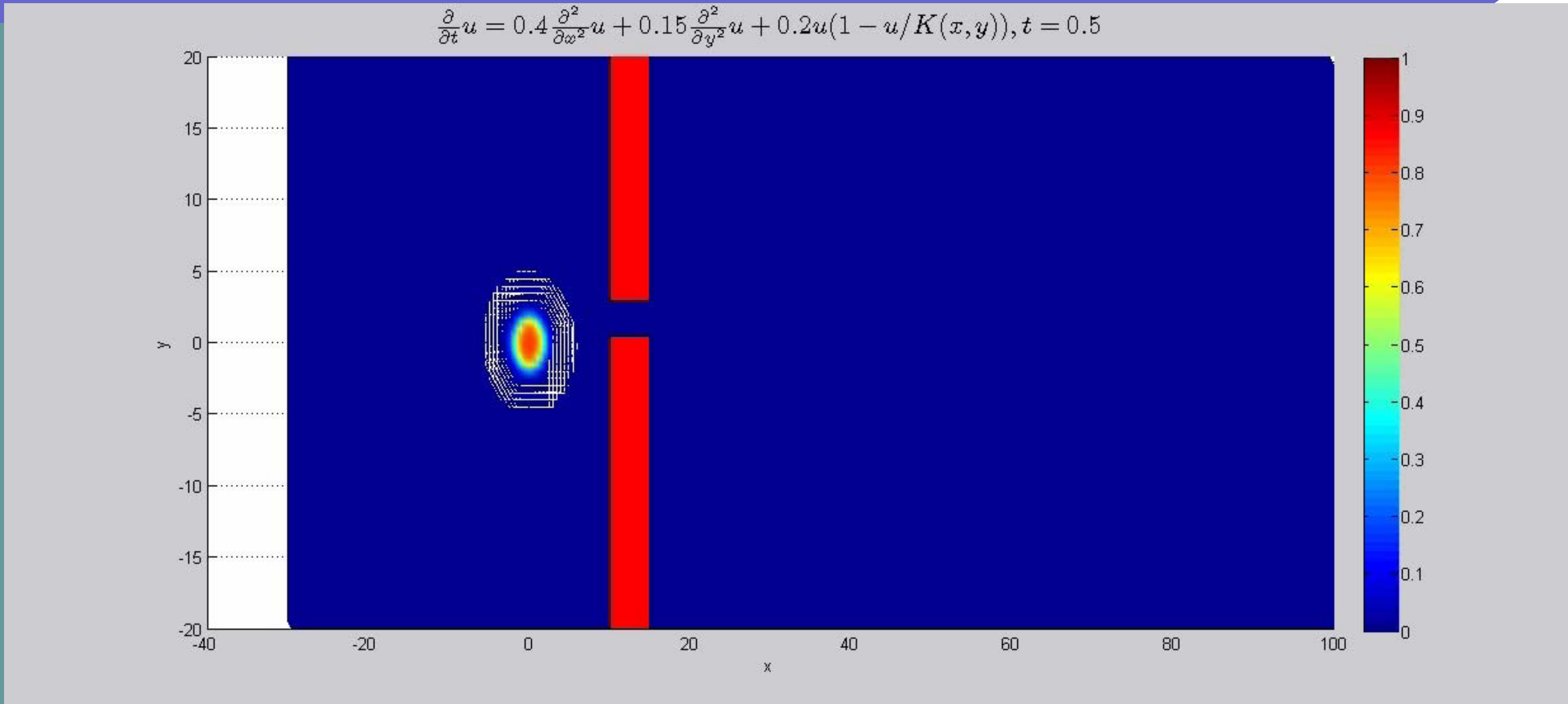
$\alpha > 2$



$\alpha < 2$

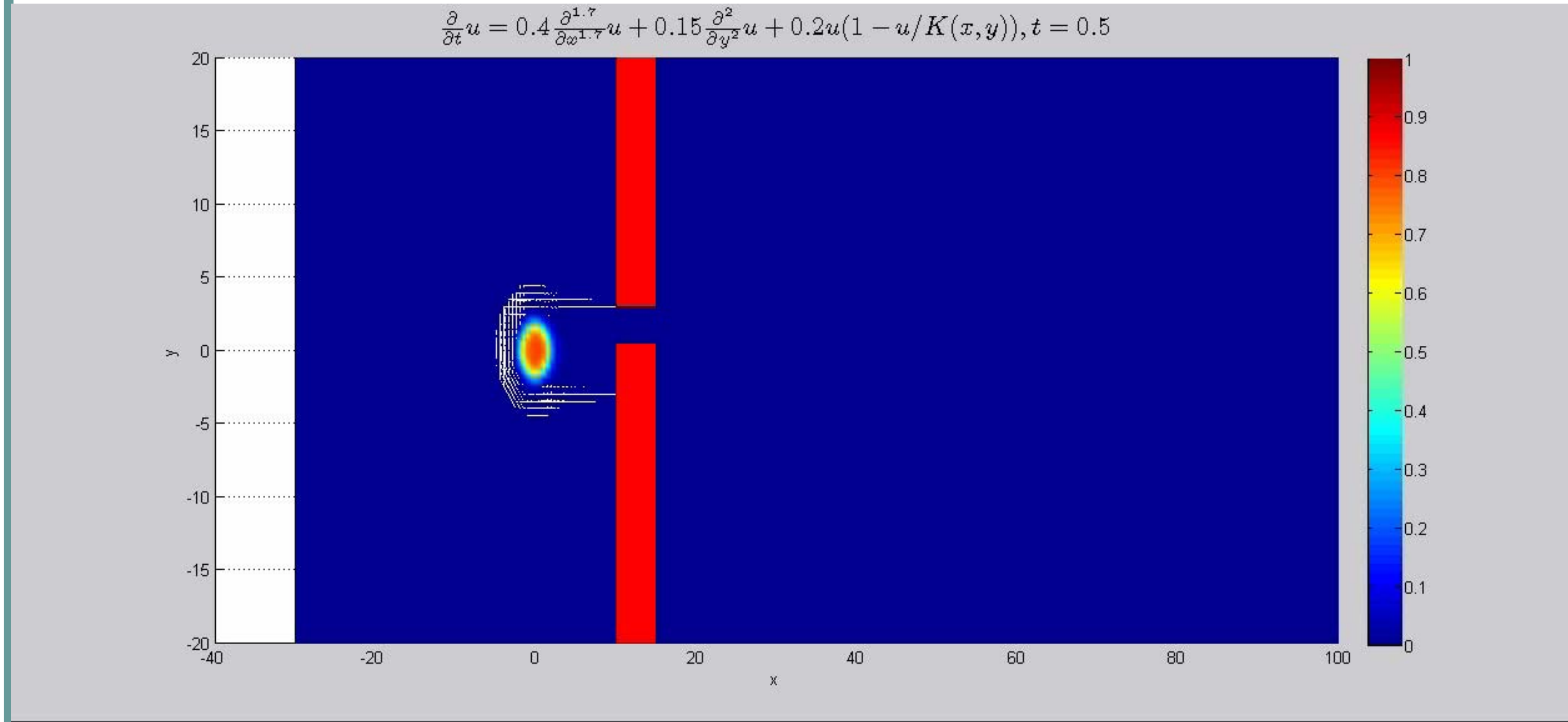


Classical Fisher equation



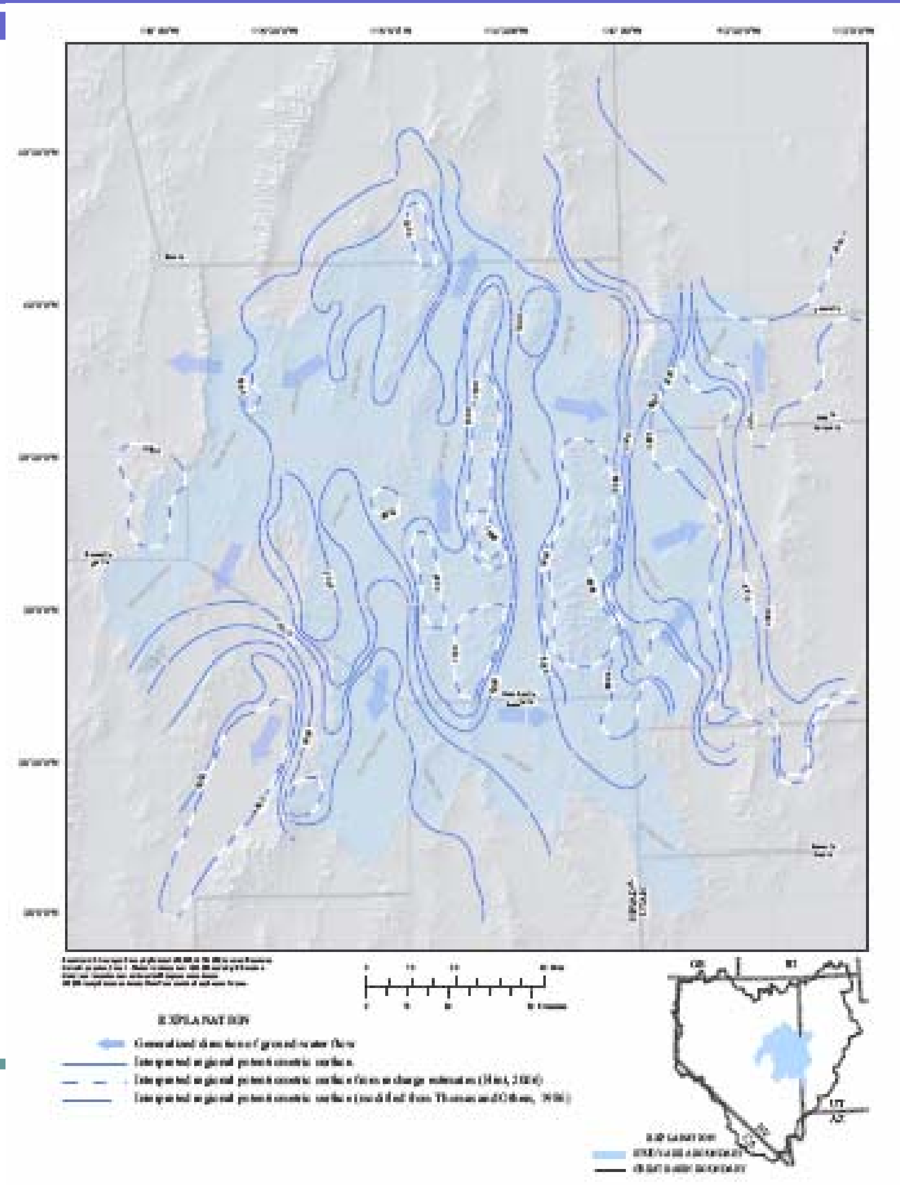
$$\frac{\partial}{\partial t}u = D\Delta u + ru(1-u/K)$$

Fractional Fisher equation



$$\frac{\partial}{\partial t}u = D_1\frac{\partial^\alpha}{\partial x^\alpha}u + D_2\frac{\partial^2}{\partial y^2}u + ru(1-u/K)$$

Movement on a regional scale



Postulates

- Long jumps should follow flow lines
- Long jump in a slow region is more improbable than in a fast region
- Locally keep one-dimensional advection and fractional dispersion with maybe some lateral diffusion.

The Idea

- Instead of jump size, randomise velocity, or time available (Bochner subordination).

$$\frac{\partial}{\partial t}u(t,\vec{x}) = -\nabla\cdot v(\vec{x})u(t,\vec{x})$$

$$C(t,x) = u(s_t,x);\ s_t \sim t + t^{1/\alpha}L(0,D)$$

- In operator semi-group notation with initial distribution *f* and flow-group *G*:

$$C(t)f = \int_{-\infty}^{\infty} g_t(s)G(s)f\,ds$$

Anomalous dispersion along flow-lines

- Main dispersal pathways (not necessarily straight) might be identifiable:
  - Randomise velocity (time) according to 1D fractional model.
  - Leads to fractional powers $1 \le \alpha \le 2$ of transport operator *A* (generator of flow group)
  - Solve by operator-split methods

$$\frac{\partial}{\partial t} u = A^{\alpha} u + ru(1 - u / K)$$

Conclusions

- Parsimonious linear model capturing (mean-field) anomalous dispersion.
- Robust as based on limit theorem.
- Extends classical models by simply replacing diffusion operator with fractional diffusion operator
- Predicts accelerating wave-speed, power-law tails, etc…

References

- B.Baeumer, M. Kovacs, and M.M. Meerschaert. Fractional reproduction-dispersal equations and heavy tail dispersal kernels. *Bulletin of Mathematical Biology,* online first *2007.* doi:10.1007/s11538-007-9220-2.
- B.Baeumer, M. Kovacs, and M.M. Meerschaert. Numerical solutions for fractional reaction-diffusion equations. To appear in *Computers and Mathematics with Applications*.
- B.Baeumer and M. Kovacs, Subordinated groups of linear operators: properties via the transference principle and the related unbounded operational calculus Submitted.